Es. 1

$n = 90$

(i)

$P\{X=k\} = 1/90$

$$P\{Y=k\} = \frac{P\{X=h, Y=k\}}{P\{X=h \mid Y=k\}} = \frac{\frac{1}{90} \cdot \frac{1}{89}}{\frac{1}{89}} = 1/90$$

X e Y indipendenti $\iff P\{X\in A, Y\in B\} = P\{X\in A\}P\{Y\in B\}$

$P\{X=h, Y=k\} = P\{X=h\}P\{Y=k\}$

$\frac{1}{90\cdot 89} \neq \frac{1}{90\cdot 90} \longrightarrow X$ e Y non sono indipendenti

(ii)

$E[X+Y] = E[X] + E[Y] = 2E[X]$

$$E[X] = \frac{\cancel{90}\cdot 91}{\cancel{90}\cdot \cancel{2}} \cdot \cancel{2} = 91$$

(iii)

$Z \sim B(10, 1/90)$ $\qquad E[X] = np = \frac{1}{9}$

$$P\{Z=1\} = \binom{10}{1}\left(\frac{1}{90}\right)\left(\frac{89}{90}\right)^9 = \frac{1}{9}\cdot\left(\frac{89}{90}\right)^9 \sim \frac{0.9}{9} \sim 0.1$$

Es. 2

(i)

$Y = X^{-1}$ $\qquad X \sim \Gamma(2,1)$ $\qquad g(x) = XY = X \cdot X^{-1} = 1$

$E[XY] = E[g(x)] = \int g(x) f(x)\, dx = \int f(x)\, dx = \Gamma(2) = 1$

$Cov(X,Y) = E[XY] - E[X]E[Y] = 1 - 2 = -1$

(ii)

Y ha momento K se $E[Y^K] < +\infty$

$$E[Y^K] = \int_0^{+\infty} \left(\frac{1}{x}\right)^k f(x)\, dx = \int_0^{+\infty} \frac{e^{-x}}{x^{k-1}}\, dx = \int_0^1 x^{1-k} e^{-x} + \underbrace{\int_1^{+\infty} x^{1-k} e^{-x}}_{<+\infty} =$$

$< +\infty \iff K < 2$

(iii)

$Y = h(X)$ $\qquad h(x) = t^{-1}$ $\qquad h'(x) = \frac{1}{y^2}$

$t^{-1} = \frac{1}{y}$

$$f_Y(y) = f_X\left(\frac{1}{y}\right) \cdot \frac{1}{y^2} = \frac{1}{y} \cdot e^{-\frac{1}{y}} \cdot \frac{1}{y^2} = \frac{e^{-\frac{1}{y}}}{y^3}$$

Es. 3

$$\int_{-\infty}^{+\infty} f(x)\, dx = 1$$

$$\int_1^{+\infty} c\, x^{-(1+\theta)}\, dx = c\int_1^{+\infty} x^{-(1+\theta)} = c\left[-\frac{x^{-\theta}}{\theta}\right]_1^{+\infty} = \frac{c}{\theta} = 1 \iff c = \theta$$

$$F(x) = \int_1^x t^{-(1+\theta)} \cdot \theta\, dt = \theta \int_1^x t^{-(1+\theta)}\, dt = \theta\left[-\frac{t^{-\theta}}{\theta}\right]_1^x = \theta\left[\frac{1}{\theta} - \frac{1}{x^{\theta}\theta}\right] =$$

$$= 1 - \frac{1}{x^{\theta}}$$

(ii)

$$E[X^2] = \int_1^{+\infty} x^2 \, \theta \, x^{-(1+\theta)} \, dx = \theta \int_1^{+\infty} x^{(1-\theta)} \, dx = \theta \left[\frac{x^{2-\theta}}{2-\theta} \right]_1^{+\infty} =$$

$$< +\infty \iff \theta > 2$$

(iii)

<u>Max Verosimiglianza</u>:

$$f_\theta(x) = \begin{cases} \theta x^{-1-\theta} \\ 0 \end{cases}$$

$$L(\theta; x_1, \dots, x_n) = \prod_{i=1}^{n} f_\theta(x_i) = \theta^n (x_1 \cdot \dots \cdot x_n)^{-1-\theta} =$$

$$\log(L) = \log(\theta)^n \cdot \log(x_1 \cdot \dots \cdot x_n)^{-1-\theta} = n \log(\theta) \cdot (1-\theta) \log(x_1 \cdots x_n)$$

$$= \nexists \, \hat{\theta}$$

- <u>Metodo dei momenti</u>:

$$E[X] = \frac{1}{n} \sum_i x_i$$

$$\int_1^{+\infty} x f(x) \, dx = \frac{1}{n} \sum_i x_i$$

$$\int_1^{+\infty} x \, \theta \, x^{-1-\theta} \, dx = \theta \int_1^{+\infty} x^{-\theta} \, dx = \theta \left[\frac{x^{1-\theta}}{1-\theta} \right]_1^{+\infty} = \frac{\theta}{\theta - 1}$$

$$\frac{\theta}{\theta-1} = \frac{1}{n} \sum_i x_i \iff \theta = \frac{(\theta-1)}{n} \sum_i x_i \iff n\theta = \theta \sum_i x_i - \sum_i x_i$$

$$\Leftrightarrow \theta - \frac{\theta}{n}\sum_i x_i = -\frac{1}{n}\sum x_i \Leftrightarrow \theta\left(1-\sum_i x_i\right) = -\frac{1}{n}\sum_i x_i$$

$$\Leftrightarrow \theta = \frac{-\frac{1}{n}\sum_i x_i}{\left(1-\sum_i x_i\right)\frac{1}{n}} \Leftrightarrow \theta = \frac{\frac{1}{n}\sum x_i}{\frac{1}{n}\left(\sum x_i - 1\right)}$$