Somma di variabili gaussiane

Date $X \sim N(m_1, \sigma_1^2)$, $Y \sim N(m_2, \sigma_2^2)$ indipendenti,

$$X + Y \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2).$$

Dimostrazione

Con la formula di convoluzione è complicato.
Sia $Z \sim N(0,1)$. Allora:

$$\begin{aligned}
G_Z(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{tx} e^{\frac{-x^2}{2}} \mathrm{d}x \quad = \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2} + tx - \frac{x^2}{2}} \mathrm{d}x \\
&= \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-t)^2} \mathrm{d}x = \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-t)^2} \mathrm{d}x \\
&\overset{y=x-t}{=} \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} \mathrm{d}y \quad = e^{\frac{t^2}{2}}
\end{aligned}$$

$$\begin{aligned}
G_X(t) &= G_{m_1 + \sigma_1 Z}(t) = \exp\left(t m_1 + \frac{\sigma_1^2 t^2}{2}\right) \\
G_Y(t) &= G_{m_2 + \sigma_2 Z}(t) = \exp\left(t m_2 + \frac{\sigma_2^2 t^2}{2}\right) \\
G_{X+Y}(t) &= \exp\left(t(m_1 + m_2) + \frac{(\sigma_1^2 + \sigma_2^2) t^2}{2}\right)
\end{aligned}$$

Che è la g.d.m. di $S \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$, quindi per il teorema $X + Y$ e S sono equidistribuite.