Densità Γ

Le variabili $\Gamma(r,\lambda)$, con $r,\lambda>0$, hanno densità:

$$f(x)=\begin{cases}\frac{1}{\Gamma(r)}\lambda^r x^{r-1}e^{-\lambda x} & x>0\\ 0 & x\leq 0\end{cases}$$

È una densità:

$$\int_0^{+\infty}\lambda^r x^{r-1}e^{-\lambda x}x\mathrm{d}x \overset{t=\lambda x}{=} \int_0^{+\infty} t^{r-1}e^{-t}\mathrm{d}t=\Gamma(r).$$

Le variabili esponenziali di parametro λ sono $\Gamma(1,\lambda)$

Formule per i momenti

$X\sim\Gamma(r,\lambda)$, $\beta\in\mathbb{R}+$

$$E[X^\beta]=\frac{\Gamma(r+\beta)}{\Gamma(r)\lambda^\beta},$$

da cui si ricava:

$$E[X]=\frac{\Gamma(r+1)}{\Gamma(r)\lambda}=\frac{r\Gamma(r)}{\Gamma(r)\lambda}=\frac{r}{\lambda}$$

$$E[X^2]=\frac{\Gamma(r+2)}{\Gamma(r)\lambda^2}=\frac{(r+1)r}{\lambda^2}$$

$$\mathrm{var}(X)=\frac{(r+1)r}{\lambda^2}-\frac{r^2}{\lambda^2}=\frac{r}{\lambda^2}$$

Dimostrazione

Ha sempre senso $E[X^\beta]$ perché $X>0$, e

$$\begin{aligned}E[X^\beta]&=\frac{1}{\Gamma(r)}\int_0^{+\infty}x^\beta\lambda^r x^{r-1}e^{-\lambda x}\mathrm{d}x\\ &=\frac{1}{\Gamma(r)\lambda^\beta}\int_0^{+\infty}\lambda^{r+\beta}x^{r+\beta-1}e^{-\lambda x}\mathrm{d}x\\ &=\frac{\Gamma(r+\beta)}{\Gamma(r)\lambda^\beta}\end{aligned}$$

Funzione generatrice dei momenti

$$G_X(t)=\begin{cases}\left(\frac{\lambda}{\lambda-t}\right)^r & t<\lambda\\ +\infty & t\geq\lambda\end{cases}$$

Dominio: $(-\infty,\lambda]$

$$\begin{aligned}G_X(t)&=\frac{1}{\Gamma(r)}\int_0^{+\infty}e^{tx}\lambda^r x^{r-1}e^{-\lambda x}\mathrm{d}x\\ &=\frac{1}{\Gamma(r)}\int_0^{+\infty}\lambda^r x^{r-1}e^{-(\lambda-t)x}\mathrm{d}x\\ &=\begin{cases}+\infty & t\geq\lambda\\ \frac{\lambda^r}{\Gamma(r)(\lambda-t)^r}\int_0^{+\infty}(\lambda-t)^r x^{r-1}e^{-(\lambda-t)r}\mathrm{d}x=\left(\frac{\lambda}{\lambda-t}\right)^r & t<\lambda\end{cases}\end{aligned}$$

$X\sim\Gamma(r_1,\lambda)$, $Y\sim\Gamma(r_1,\lambda)$ indipendenti, allora:

$$(X+Y)\sim\Gamma(r_1+r_2,\lambda).$$

Infatti

$$G_{X+Y}(t)=G_X(t)G_Y(t)=\left(\frac{\lambda}{\lambda-t}\right)^{r_1}\left(\frac{\lambda}{\lambda-t}\right)^{r_2}=\left(\frac{\lambda}{\lambda-t}\right)^{r_1+r_2}.$$